ON THE MAXIMUM DRAWDOWN 
OF A BROWNIAN MOTION

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Abstract

The maximum drawdown at time $T$ of a random process on $[0, T]$ can be defined
informally as the largest drop from a peak to a trough. In this paper, we investigate
the behaviour of this statistic for a Brownian motion with drift. In particular, we give
an infinite series representation of its distribution and consider its expected value. When
the drift is zero, we give an analytic expression for the expected value, and for nonzero
drift, we give an infinite series representation. For all cases, we compute the limiting
($T \to \infty$) behaviour, which can be logarithmic (for positive drift), square root (for zero
drift) or linear (for negative drift).

Keywords: Random walk; asymptotic distribution; expected maximum drawdown

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1. Introduction

The maximum drawdown is commonly used in finance as a measure of risk for a stock that
follows a particular random process. Here we consider the maximum drawdown of a Brownian
motion. Let $W(t)$, $0 \leq t \leq T$, be a standard Wiener process and let $X(t)$ be the Brownian
motion given by $X(t) = \sigma W(t) + \mu t$, where $\mu \in \mathbb{R}$ is the drift and $\sigma \geq 0$ is the diffusion
parameter. The high $H$, the low $L$ and the range $R$ of $X$ are defined by

$$H = \sup_{t \in [0,T]} X(t), \quad L = \inf_{t \in [0,T]} X(t), \quad R = H - L.$$ 

The maximum drawdown is defined by

$$\tilde{D} = \tilde{D}(T; \mu, \sigma) = \sup_{t \in [0,T]} \left[ \sup_{s \in [0,t]} X(s) - X(t) \right].$$

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Denote the distribution function for \( \tilde{D} \) by \( G_{\tilde{D}}(h) = P[\tilde{D} \geq h] \). We will show that

\[
G_{\tilde{D}}(h) = 2\sigma^2 \sum_{n=1}^{\infty} \frac{\theta_n \sin \theta_n}{\sigma^4 \theta_n^2 + \mu^2 h^2 - \sigma^2 \mu h} \exp \left( -\frac{\mu h}{\sigma^2} \right) \left( 1 - \exp \left( -\frac{\sigma^2 \theta_n^2 T}{2h^2} \right) \exp \left( \frac{\mu^2 T}{2\sigma^2} \right) \right) + M, \tag{1}
\]

where, for \( n \geq 1 \), \( \theta_n \) is the positive solution of the eigenvalue condition

\[
\tan \theta_n = \frac{\sigma^2}{\mu h} \theta_n \tag{2}
\]

and \( M \) is given by

\[
M = \begin{cases} 
0, & \mu < \frac{\sigma^2}{h}, \\
\frac{3}{e} \left( 1 - \exp \left( -\frac{\mu^2 T}{2\sigma^2} \right) \right), & \mu = \frac{\sigma^2}{h}, \\
\frac{2\sigma^4 \eta \sinh \eta \exp \left( -\mu h / \sigma^2 \right)}{\sigma^4 \eta^2 - \mu^2 h^2 + \sigma^2 \mu h} \left( 1 - \exp \left( -\frac{\mu^2 T}{2\sigma^2} \right) \exp \left( \frac{\sigma^2 \eta^2 T}{2h^2} \right) \right), & \mu > \frac{\sigma^2}{h},
\end{cases} \tag{3}
\]

where \( \eta \) is the unique positive solution of

\[
tanh \eta = \frac{\sigma^2}{\mu h}. \tag{4}
\]

Using the identity \( E[\tilde{D}] = \int_0^\infty dh \ G_{\tilde{D}}(h) \) and defining \( \alpha = \mu \sqrt{T/2\sigma^2} \), we find that

\[
E[\tilde{D}] = \frac{2\sigma^2}{\mu} Q_{\tilde{D}}(\alpha^2), \tag{5}
\]

where

\[
Q_{\tilde{D}}(x) = \begin{cases} 
Q_p(x), & \mu > 0, \\
\frac{\gamma}{2} \sqrt{2x}, & \mu = 0, \\
-Q_n(x), & \mu < 0,
\end{cases}
\]

with \( \gamma = \sqrt{\frac{\pi}{8}} \approx 0.6267 \) a constant, and \( Q_p \) and \( Q_n \) functions whose exact expressions are given in (12) and (10) respectively. These functions can be evaluated numerically (a comprehensive list of values is given in Table 1 below), and their asymptotic behaviour is given by

\[
Q_p(x) \rightarrow \begin{cases} 
\frac{\gamma}{2} \sqrt{2x}, & x \rightarrow 0^+, \\
\frac{1}{4} \log x + 0.49088, & x \rightarrow \infty,
\end{cases}
\]

\[
Q_n(x) \rightarrow \begin{cases} 
\frac{\gamma}{2} \sqrt{2x}, & x \rightarrow 0^+, \\
x + \frac{1}{2}, & x \rightarrow \infty.
\end{cases}
\]

The asymptotic behaviour is logarithmic for \( \mu > 0 \), linear for \( \mu < 0 \) and square root for \( \mu = 0 \). A similar result in the asymptotic case was obtained by Berger and Whitt [1]. For positive \( \mu \),
using a reflected Brownian motion [8], they considered the asymptotic distribution of queueing processes and obtained a Gumbel distribution, which is consistent with our findings. Douady et al. [5] considered the maximum drawdown for a Brownian motion when the drift is zero and obtained similar results to ours in this special case. The behaviour of $Q_D(\cdot)$ is shown in Figure 1.

2. Derivations

Let $D(t), t \in [0, T]$, be the random process defined by

$$D(t) = \sup_{s \in [0, t]} X(s) - X(t).$$

Then $D(t)$ is the drawdown from the previous maximum value at time $t$. It is well known that $D(t)$ is a reflected Brownian motion on $[0, T]$,

$$dD(t) = \begin{cases} -dX(t), & D(t) > 0, \\ \max\{0, -dX(t)\}, & D(t) = 0. \end{cases}$$

(If $X(t)$ has drift and diffusion parameters $\mu_X$ and $\sigma$, then $D(t)$ is a Brownian motion with drift $-\mu_X$ and diffusion parameter $\sigma$; $D(0) = 0$ and $D(t)$ has a reflective barrier at 0.) A rigorous justification of (6) can be found in [8]. From $D(t)$, we can get the maximum drawdown, $\bar{D} = \sup_{t \leq T} D(t)$. Let $h > 0$ be an absorbing barrier, let $\tau$ be the absorption time and let $f_\tau$ be the absorption time density, i.e. $f_\tau(t \mid h)dt$ is the probability that $\tau \in [t, t + dt]$. Let $G_{\bar{D}}(h)$ be the probability that $\bar{D} \geq h$ in the interval $[0, T]$. Then

$$G_{\bar{D}}(h) = \int_0^T dt \ f_\tau(t \mid h).$$

(7)
The absorption time density $f_t(t \mid h)$ can be computed from a more general result for a Brownian motion between two partly absorbing barriers given in [4], and is given by

$$f_t(t \mid h) = \exp \left\{-\frac{\mu^2 t}{2\sigma^2}\right\} \times \frac{\sigma^2}{h^2} \sum_{n=0}^{\infty} \frac{(\sigma^4 \theta_n^2 + \mu^2 h^2) \delta_n \sin \theta_n}{\sigma^4 \theta_n^2 + \mu^2 h^2 - \sigma^2 \mu h} \exp \left\{-\frac{\mu h}{\sigma^2}\right\} \exp \left\{-\frac{\sigma^2 \theta_n^2 t}{2h^2}\right\} + K,$$  \hspace{1cm} (8)

where $\theta_n$ is the positive solution to the eigenvalue condition (2), $K$ is given by

$$K = \begin{cases} 0, & \mu < \frac{\sigma^2}{h}, \\ \frac{3\sigma^2}{2eh^2}, & \mu = \frac{\sigma^2}{h}, \\ \frac{\sigma^2}{h^2 \sigma^4 \eta^4 - \mu^2 h^2 + \sigma^2 \mu h} \sinh \eta \exp \left\{-\frac{\mu h}{\sigma^2}\right\} \exp \left\{\frac{\sigma^2 \eta^2 t}{2h^2}\right\}, & \mu > \frac{\sigma^2}{h}, \end{cases}$$  \hspace{1cm} (9)

and $\eta$ is the unique positive solution to the eigenvalue condition (4). Alternatively, in Section 3, we will obtain the same result by taking the continuous limit of the discrete random walk. Substituting (8) into (7), and after an integration, we arrive at (1). To get the expectation (5), we use the identity $\text{E}[\bar{D}] = \int_0^\infty dh \, G_{\bar{D}}(h)$, which is valid for positive-valued random variables.

2.1. Case $\mu = 0$

In this case, the eigenvalue condition (2) is solved by $\theta_n = (n - \frac{1}{2})\pi$. Thus,

$$G_{\bar{D}}(h) = 2 \sum_{n=1}^{\infty} \frac{\sin((n - \frac{1}{2})\pi)}{(n - \frac{1}{2})\pi} \left(1 - \exp \left\{-\frac{\sigma^2 (n - \frac{1}{2})^2 \pi^2 T}{2h^2}\right\}\right)$$

$$= \frac{2}{\pi} g\left(\frac{h}{\pi \sigma \sqrt{T}}\right),$$

where

$$g(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n + \frac{1}{2}} \left(1 - \exp\left\{-\frac{(n + \frac{1}{2})^2}{2x^2}\right\}\right).$$

The expected value of $\bar{D}$ is then given by

$$\text{E}[\bar{D}] = \int_0^\infty dh \, G_{\bar{D}}(h)$$

$$= \frac{2}{\pi} \int_0^\infty dh \, g\left(\frac{h}{\pi \sigma \sqrt{T}}\right)$$

$$= 2\gamma \sigma \sqrt{T},$$

where

$$\gamma = \int_0^\infty dh \, g(h) = \sqrt{\pi}/8 \approx 0.6267.$$
On the maximum drawdown of a Brownian motion

The exact computation of $\gamma$ from the integral

$$\int_0^\infty dh \ g(h)$$

is challenging. An alternate route to the same expression using reflected Brownian motion was obtained by Bond [3], who showed that $\gamma = \sqrt{\pi}/8$. A useful comparison is to the expected value of the range $R$, the difference between the two extrema of the motion. The range $R$ is computed for $\mu = 0$ in [6], and the generalization to nonzero $\mu$ can be obtained from (29) in Appendix A and the identity $E[R] = E[H \mid \mu] + E[H \mid -\mu]$, where $H$ is the maximum of the motion. For $\mu = 0$, $E[R] = 2\sqrt{2/\pi \alpha \sqrt{\bar{T}}}$ and $\sqrt{2/\pi} \approx 0.798 > \gamma$; thus the expected range is considerably larger than the expected maximum drawdown for $\mu = 0$.

2.2. Case $\mu < 0$

After applying the eigenvalue conditions and taking the integral of $G_{\mu}(h)$ to get the expectation, we find that

$$E[\bar{D}] = \int_0^\infty dh \ G_{\mu}(h)$$

$$= 2 \int_0^\infty dh \ \exp\left\{ -\frac{\mu h}{\sigma^2} \right\} \sum_{n=1}^\infty \frac{\sin^3 \theta_n}{\theta_n - \cos \theta_n \sin \theta_n} \left( 1 - \exp\left\{ -\frac{\mu^2 T}{2\sigma^2 \cos^2 \theta_n} \right\} \right).$$

Making a change of variables to $u = -\mu h/\sigma^2$, we find that

$$E[\bar{D}] = -2 \frac{\sigma^2}{\mu} Q_n(\alpha^2),$$

where, for $x > 0$,

$$Q_n(x) = \int_0^\infty du \ e^u \sum_{n=1}^\infty \frac{\sin^3 \theta_n}{\theta_n - \cos \theta_n \sin \theta_n} \left( 1 - \exp\left\{ -\frac{x}{\cos^2 \theta_n} \right\} \right). \quad (10)$$

Here, each $\theta_n$ satisfies $\tan \theta_n = -\theta_n/\mu$ and $\alpha = \mu \sqrt{T/2\sigma^2}$. The numerical computation of $Q_n(x)$ is not a straightforward task. The sum in the integrand is a function of $u$ that decreases faster than $e^{-u}$. Since the magnitude of the $n$th term in the sum is approximately $1/n$, we need to take $\Omega(e^n)$ terms in the sum to make sure that the next term left out has magnitude less than the size of the sum. Thus, efficient computation of this integral is computationally nontrivial. Table 1 gives approximate values of $Q_n(x)$ for various values of $x$, computed using an extensive numerical integration. Intermediate values can be obtained using interpolation and the asymptotic behaviour is discussed below.

When $x \to 0^+$, $Q_n(x) \to \gamma \sqrt{2x}$ since, in this limit, we must recover the behaviour as $\mu \to 0$. We get the behaviour in the limit as $x \to \infty$ by noting that $R \geq \bar{D} \geq -L$. Taking expectations and using (29), we see that, for all $\alpha$,

$$\frac{\alpha^2}{2} + \frac{Q_R(-\alpha)}{2} \leq Q_n(\alpha^2) \leq Q_R(-\alpha). \quad (11)$$

where

$$Q_R(x) = \text{erf}(x)(\frac{1}{2} + x^2) + \frac{1}{\sqrt{\pi}} xe^{-x^2} \quad \text{and} \quad \text{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x du \ e^{-u^2}.$$
Table 1: Numerical values for $Q_p(\cdot)$ (for positive $\mu$) and $Q_n(\cdot)$ (for negative $\mu$).

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<th>$Q_p(x)$, $\mu &gt; 0$</th>
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$x \to \infty \frac{1}{2} \log x + 0.49088$
Asymptotically, as $\alpha \to -\infty$, this yields that

$$a^2 + \frac{1}{4} \leq Q_n(a^2) \leq a^2 + \frac{1}{2},$$

from which we deduce that $Q_n(x) \to x + \varepsilon(x)$, where $\frac{1}{4} \leq \varepsilon(x) \leq \frac{1}{2}$. We now argue that, asymptotically, $E[\bar{D}] = E[R]$ when $\mu < 0$. By the definition of $\bar{D}$, when the maximum occurs before the minimum, $\bar{D} = R$, so

$$E[\bar{D}] \geq E[R \mid H \to L] \cdot P[H \to L],$$

where $A \to B$ denotes the event ‘$A$ occurs before $B$’. Since $\mu < 0$, $P[H \to L] \to 1$ as $T \to \infty$. Considering the range, we have

$$E[R] = E[R \mid H \to L] \cdot P[H \to L] + E[R \mid L \to H] \cdot P[L \to H].$$

The term $E[R \mid L \to H]$ is slowly growing with $T$, but, for $\mu < 0$, $P[L \to H] \to 0$ exponentially fast, hence the second term asymptotically approaches 0, and so we conclude that $E[R] \to E[R \mid H \to L]$. Thus, we see that, asymptotically, $E[\bar{D}] \geq E[R]$, hence that $E[\bar{D}] = E[R]$, and so $\lim_{x \to \infty} \varepsilon(x) = \frac{1}{2}$. Collecting the results together, we have

$$Q_n(x) \to \begin{cases} \gamma \sqrt{2x}, & x \to 0^+, \\ x + \frac{1}{2}, & x \to \infty. \end{cases}$$

The asymptotic behaviour is illustrated in Figure 2, which shows the numerical computation of $Q_n(\cdot)$ along with the asymptote.
2.3. Case $\mu > 0$

In this case, for $h > \sigma^2/\mu$ in the integral (7), the expression for $M$ in (3) adds another term. Thus, we find that

$$E[\tilde{D}] = \int_{0}^{\infty} dh \, G_{\tilde{D}}(h),$$

$$= 2 \int_{0}^{\infty} dh \, \exp \left\{ -\frac{\mu h}{\sigma^2} \right\} \sum_{n=1}^{\infty} \frac{\sin^3 \theta_n}{\theta_n - \cos \theta_n \sin \theta_n} \left( 1 - \exp \left\{ -\frac{\mu^2 T}{2\sigma^2 \cos^2 \theta_n} \right\} \right)$$

$$- 2 \int_{\sigma^2/\mu}^{\infty} dh \, \exp \left\{ -\frac{\mu h}{\sigma^2} \right\} \frac{\sinh^3 \eta}{\eta - \cosh \eta \sinh \eta} \left( 1 - \exp \left\{ -\frac{\mu^2 T}{2\sigma^2 \cosh^2 \eta} \right\} \right).$$

The second integral can be reduced by a change of variables $u = \eta(h)$ as follows. Since $\tanh u = \sigma^2 u / \mu h$,

$$\frac{dh}{du} = \frac{\sigma^2}{\mu} \frac{\cosh u \sinh u - u}{\sinh^2 u},$$

hence, the second integral reduces to

$$- \frac{\sigma^2}{\mu} \int_{0}^{\infty} du \exp \left\{ -\frac{u}{\tanh u} \right\} \sinh u \left( 1 - \exp \left\{ -\frac{\mu^2 T}{2\sigma^2 \cosh^2 u} \right\} \right).$$

Changing variable in the first integral to $u = \mu h / \sigma^2$, we arrive at

$$E[\tilde{D}] = \frac{2\sigma^2}{\mu} Q_p(\alpha^2),$$

where $\alpha = \mu \sqrt{T} / 2\sigma^2$ once again and, for $x > 0$,

$$Q_p(x) = \int_{0}^{\infty} du \left[ e^{-u} \sum_{n=0}^{\infty} \frac{\sin^3 \theta_n}{\theta_n - \cos \theta_n \sin \theta_n} \left( 1 - \exp \left\{ -\frac{x}{\cos^2 \theta_n} \right\} \right) \right. $$

$$+ \left. \exp \left\{ -\frac{u}{\tanh u} \right\} \sinh u \left( 1 - \exp \left\{ -\frac{x}{\cosh^2 u} \right\} \right) \right].$$

Here, each $\theta_n$ satisfies $\tan \theta_n = \theta_n / u$. The bound (11) is still valid, but not very useful. The numerical computation of $Q_p(\cdot)$ is relatively straightforward, as the $e^{-u}$ term in the integrand makes it well behaved for the purposes of numerical integration. We know that $Q_p(x) \to \gamma \sqrt{2x}$ when $x \to 0^+$. We now consider the other asymptotic limit, namely $\alpha \to \infty$. We will evaluate the two contributions to $Q_p(\cdot)$ separately. Consider the first part of the integral in (12), which we denote by $I_1(x)$,

$$I_1(x) = \int_{0}^{\infty} du \, e^{-u} \sum_{n=0}^{\infty} \frac{\sin^3 \theta_n}{\theta_n - \cos \theta_n \sin \theta_n} \left( 1 - \exp \left\{ -\frac{x}{\cos^2 \theta_n} \right\} \right).$$

Since $0 \leq \cos^2 \theta_n \leq 1$ and $x \to \infty$, the term in parentheses is rapidly approaching 1. Since $e^{-u}$ is rapidly decreasing, we interchange the summation with the integration and, after changing variables in the integral to $v = \theta_n / u$ and using the identity

$$du = \frac{\cos v \sin u - v}{\sin^2 u} \, dv,$$
we find that
\[
I_1(x) = \sum_{n=0}^{\infty} \int_{n\pi}^{n\pi+\pi/2} dv \exp\left(-\frac{v}{\tan v}\right) \sin v \left(1 - \exp\left(-\frac{x}{\cos^2 v}\right)\right).
\]

After translating each integral by \(n\pi\) and bringing the summation back into the integral, the sum is a geometric progression, which implies that
\[
I_1(x) = \int_0^{\pi/2} dv \frac{\exp(-v/\tan v) \sin v (1 - \exp(-x/\cos^2 v))}{1 + \exp(-\pi/\tan v)}.
\]

Thus, \((1 - e^{-x})\beta_1 \leq I_1(x) \leq \beta_1\), where
\[
\beta_1 = \int_0^{\pi/2} dv \frac{\exp(-v/\tan v) \sin v}{1 + \exp(-\pi/\tan v)},
\]
and so we see that \(I_1(x)\) rapidly converges to \(\beta_1\). The constant \(\beta_1\) can be evaluated numerically to give \(\beta_1 = 0.4575\). Now consider the second term in (12), which we denote by \(I_2(x)\),
\[
I_2(x) = \int_0^{\infty} du \exp\left(-\frac{u}{\tanh u}\right) \sinh u \left(1 - \exp\left(-\frac{x}{\cosh^2 u}\right)\right).
\]

When \(x\) is large, the term in parentheses is very close to 1 until \(u\) gets so large that \(\cosh u \sim x\), from which point this term rapidly decreases to 0. The first term in the integrand is always less than \(\frac{1}{2}\) and rapidly increases from 0 to \(\frac{1}{2}\). Thus, we can write
\[
I_2(x) = \int_0^{\infty} du \left(\exp\left(-\frac{u}{\tanh u}\right) \sinh u - \frac{1}{2} + \frac{1}{2}\right) \left(1 - \exp\left(-\frac{x}{\cosh^2 u}\right)\right)
= \frac{1}{2} \int_0^{\infty} du \left(1 - \exp\left(-\frac{x}{\cosh^2 u}\right)\right) \sinh u
+ \int_0^{\infty} du \exp\left(-\frac{x}{\cosh^2 u}\right) \left(1 - \frac{1}{2} - \exp\left(-\frac{u}{\tanh u}\right) \sinh u\right).
\]

We first show that the third integral approaches zero as \(x \to \infty\). Since this integral is monotonically decreasing in \(x\), it suffices to consider the integral for \(x \in \mathbb{N}\). Let
\[
g(u) = \frac{1}{2} - \exp\left(-\frac{u}{\tanh u}\right) \sinh u
\]
and let
\[
f_n(u) = g(u) \exp\left(-\frac{n}{\cosh^2 u}\right).
\]

Then \(|f_n| < g\) almost everywhere and \(f_n \to 0\) almost everywhere. Therefore, by the Lebesgue dominated convergence theorem, the third integral in (14) converges to 0. The second integral is a constant, denoted by \(\beta_2\), independent of \(x\) and can be evaluated numerically to give \(\beta_2 = 0.4575\), which apparently is (numerically) equal to \(\beta_1\). The authors suspect that \(\beta_2 = \beta_1\) but the proof has been elusive.
We get bounds for the first integral using the inequalities \( \cosh u \geq \frac{1}{2} e^u \) and, for \( u \geq A \), \( \cosh u \leq \frac{1}{2} e^{\lambda(A)u} \) where \( \lambda(A) = 1 + e^{-2A}/A \). Denoting the first integral in (14) by \( F(x) \), we immediately get the following bounds, which hold for arbitrary fixed \( A \):

\[
A \left( 1 - \exp \left( -\frac{x}{\cosh^2 A} \right) \right) + \int_A^\infty du \left( 1 - \exp \left( -4xe^{-2\lambda(A)u} \right) \right) 
\leq 2F(x) \leq \int_0^\infty du \left( 1 - \exp \left( -4xe^{-2u} \right) \right).
\]

A change of variables to \( v = xe^{-2\lambda(A)u} \) in the lower bound and \( v = xe^{-2u} \) in the upper bound then leads to the following bounds:

\[
A \left( 1 - \exp \left( -\frac{x}{\cosh^2 A} \right) \right) + \frac{1}{2\lambda} \int_0^{xe^{-2\lambda A}} \frac{du}{u} \left( 1 - e^{-4u} \right) 
\leq 2F(x) \leq \frac{1}{2} \int_0^x \frac{du}{u} \left( 1 - e^{-4u} \right). \tag{15}
\]

We can get an asymptotic form as follows. Suppose that \( z > 1 \). Then

\[
\int_0^z \frac{du}{u} \left( 1 - e^{-4u} \right) = \int_0^1 \frac{du}{u} \left( 1 - e^{-4u} \right) + \log z - \int_1^z \frac{du}{u} e^{-4u}.
\]

The first term is a constant and, when \( z \to \infty \), the third term converges to \( \text{Ei}(-4) \) (see for example [7]). Thus, as \( z \to \infty \),

\[
\int_0^z \frac{du}{u} \left( 1 - e^{-4u} \right) \to \log z + C, \tag{16}
\]

where

\[
C = \int_0^1 \frac{du}{u} \left( 1 - e^{-4u} \right) + \text{Ei}(-4) \approx 1.9635.
\]

Substituting (16) into (15), we find that, for fixed \( A \),

\[
\frac{1}{2\lambda(A)} (\log x + C) - A \exp \left( -\frac{x}{\cosh^2 A} \right) \leq 2F(x) \leq \frac{1}{2} \log x + \frac{C}{2}.
\]

Since \( A \) is arbitrary, it can be chosen to grow with \( x \), for example \( \frac{1}{2}(1 + \varepsilon) \log x \), in which case \( \lambda(A) \to 1 \) and the second term in the lower bound approaches 0, so the upper and lower bounds approach each other. Thus, we can conclude that, as \( x \to \infty \),

\[
F(x) \to \frac{1}{2} \log x + D,
\]

where \( D = C/4 \). Collecting the results, and remembering that \( Q_p(x) = I_1(x) + I_2(x) \), we have

\[
Q_p(x) \to \begin{cases} \gamma \sqrt{2x}, & x \to 0^+, \\ \frac{1}{4} \log x + D, & x \to \infty, \end{cases}
\]

where \( D \approx 0.49088 \), and we have used the fact that \( \beta_1 \approx \beta_2 \). The asymptotic behaviour is illustrated in Figure 3.
3. Discrete random walk

We model the Brownian motion as a discrete random walk at the times \( t_i = i\Delta t \), where \( \Delta t = T/n \) and \( i = 0, 1, \ldots, n \). Define \( X_i = X(t_i) \), the position of the random walk at time \( t_i \). We assume that \( X_i \) has the following dynamics:

\[
X_{i+1} = \begin{cases} 
X_i + \delta & \text{with probability } p, \\
X_i - \delta & \text{with probability } q = 1 - p.
\end{cases}
\]

Defining

\[
\delta = \sqrt{\sigma^2 \Delta t + \mu^2 \Delta t^2},
\]

\[
p = \frac{1}{2} \left( 1 + \frac{\mu \sqrt{\Delta t}}{\sqrt{\sigma^2 + \mu^2 \Delta t}} \right),
\]

\[
q = \frac{1}{2} \left( 1 - \frac{\mu \sqrt{\Delta t}}{\sqrt{\sigma^2 + \mu^2 \Delta t}} \right),
\]

in the limit \( \Delta t \to 0 \), the random walk converges in distribution to a Brownian motion on \([0, T]\) with drift \( \mu \) and diffusion parameter \( \sigma \).

Analogously to \( D(t) \) in Section 2, define \( D_t \) to be the drawdown from the previous maximum at time-step \( t \), with \( D_0 = 0 \). The maximum drawdown is given by \( \bar{D} = \max_{i} D_t \). The drawdown \( D_t \) is a random walk with probability \( \bar{p} = 1 - p \) and a reflective barrier at 0. Let \( h > 0 \) be an absorbing barrier and let \( f(i \mid h) \) be the probability that the random walk gets absorbed at time-step \( i \). Then

\[
P(\bar{D} > h) = P[\text{absorption in } [0, T]] = \sum_{i=0}^{T/\Delta t} f(i \mid h).
\]
The probability $f(i \mid h)$ was initially computed in [9] for $p/q < (1 + 1/N)^2$. The more general case was given in [2], which, after the correction of some typographic errors, is given by

$$
 f(i \mid h) = \begin{cases} 
 f(1), & \frac{p}{q} < \left(1 + \frac{1}{N}\right)^2, \\
 \frac{3}{2} \frac{2^i p^{(1/2)(i-N)} q^{(1/2)(i+N)}}{(N + 1)(N + \frac{1}{2})}, & \frac{p}{q} = \left(1 + \frac{1}{N}\right)^2, \\
 \frac{2^i p^{1/2(i-N)} q^{1/2(i+N)} N^{1/2} \sinh^2 \beta^2}{(N + 1)q^{1/2} \cosh(N + 1) \beta - N^{1/2} \cosh N \beta}, & \frac{p}{q} > \left(1 + \frac{1}{N}\right)^2, 
\end{cases}
$$

where $N = h/\delta$ and

$$
 \tilde{f}(k) = -2^i p^{1/2(i-N)} q^{1/2(i+N)} \sum_{v=k}^{N} \frac{q^{1/2} \cos^{i-1} \alpha_v \sin^2 \alpha_v}{(N + 1) \cos(N + 1) \alpha_v - N^{1/2} \cos N \alpha_v},
$$

with $\alpha_v \in (v\pi/(N - 1), (v + 1)\pi/(N - 1))$ satisfying

$$
 q^{1/2} \sin(N + 1) \alpha_v - p^{1/2} \sin N \alpha_v = 0,
$$

and $\beta$ satisfying

$$
 q^{1/2} \sinh(N + 1) \beta - p^{1/2} \sinh N \beta = 0.
$$

Using (19) and (20) in (18) gives the distribution function in the discrete case.

Using $\delta$ and $p$ as given in (17), and taking the limit as $\Delta t \to 0$ gives the continuous case as follows. The sum in (18) is the Riemann sum approximation to the integral $\int_0^T \frac{1}{\Delta t} f_i(t \mid h)$, where $f_i(t \mid h)$ is the absorption time density given by the limit of $f(i \mid h)/\Delta t$ where $i \Delta t = t$. It thus remains to take the limit as $\Delta t \to 0$ of $f(i \mid h)/\Delta t$. Since $p = \frac{1}{2}(1 + \lambda)$, where $\lambda = \mu/\sigma$ and $\delta = \sigma/\Delta t$, the three cases in (19) reduce to $\mu < \sigma^2/h$, $\mu = \sigma^2/h$ and $\mu > \sigma^2/h$ in the limit, analogous to the three cases in (9). Using the identity $\lim_{x \to \infty} (1 + 1/x)^2 = e$, we find that

$$
 2^i p^{1/2(i-N)} q^{1/2(i+N)} = (1 - \lambda^2)^{i/2} \left(\frac{1 - \lambda}{1 + \lambda}\right)^{N/2} \exp\left\{-\frac{\mu^2 t}{2\sigma^2}\right\} \exp\left\{-\frac{\mu h}{\sigma^2}\right\}.
$$

Expanding the eigenvalue condition for $\alpha_v$ in (21) to first order in $\lambda$ and using some trigonometric identities gives

$$
 \tan(N + \frac{1}{2}) \alpha_v \cos \alpha_v = \frac{\sin \alpha_v}{\lambda}.
$$

Since $\alpha_v \in (v\pi/(N - 1), (v + 1)\pi/(N - 1))$, let $\theta_v = (N + \frac{1}{2}) \alpha_v$. For fixed $v$, $\alpha_v \to 0$, so we take the first-order expansion in $\alpha_v$ to get

$$
 \tan \theta_v = \frac{\sigma^2}{\mu h} \theta_v,
$$

with

$$
 \theta_v \in \left(\frac{v\pi}{N - 1}, \frac{(v + 1)\pi}{N - 1}\right) \to (v\pi, (v + 1)\pi).
$$
In an identical manner, we can analyse the eigenvalue condition for $\beta$ to get
\[ \tanh(N + \frac{1}{2})\beta \cosh\beta = \frac{2}{\lambda} \sinh\frac{\beta}{2}. \]

Defining $\eta = (N + \frac{1}{2})\beta$ and taking the limit, we find that
\[ \tanh \eta = \frac{\sigma^2}{\mu h} \eta. \]

We now analyse the summands in the expression (20) for $\tilde{f}$. Since $\sin \alpha_v \to \theta_v/(N + \frac{1}{2})$, these summands become
\[ \frac{\theta_v^2 \cos^{-1} \alpha_v}{(N + \frac{1}{2})(N + 1)[\cos(\theta_v + \frac{1}{2}\alpha_v) - A \cos(\theta_v - \frac{1}{2}\alpha_v)]}, \tag{24} \]

where $A = (1 + \lambda)^{1/2}/(1 + 1/N)(1 - \lambda)^{1/2}$. The identity $\lim_{x \to 0} \cos^{1/2} x = e^{-1/2}$ implies that $\cos^{-1} \alpha_v \to \exp(-\sigma^2 \theta_v^2/2h^2)$. Thus, after using some double-angle formulae and taking the limit, the summands in $\tilde{f}$ become
\[ \frac{\Delta t \sigma^2}{h^2} \frac{\theta_v^2 \exp(-\sigma^2 \theta_v^2/2h^2)}{(1 - \mu h/\sigma^2) \cos \theta_v - \theta_v \sin \theta_v} = \frac{-\theta_v \sin \theta_v \theta_v [\sigma^4 \theta_v^2 + \mu^2 h^2]}{\sigma^4 \theta_v^2 + \mu^2 h^2 - \mu h \sigma^2}, \tag{25} \]

where the second equality follows by use of (23) and some trigonometric identities. Substituting all these results back into (20) and dividing by $\Delta t$, we finally arrive at the continuous limit of $\tilde{f}$:
\[ \tilde{f}(t) \to \exp\left\{-\frac{\mu^2 t}{2 \sigma^2}\right\} \exp\left\{-\frac{\mu h}{\sigma^2}\right\} \frac{\theta_v^2}{h^2} \sum_{v=0}^{\infty} \theta_v \sin \theta_v \theta_v [\sigma^4 \theta_v^2 + \mu^2 h^2] \exp(-\sigma^2 \theta_v^2/2h^2), \tag{26} \]

where each $\theta_v$ is a positive solution to the eigenvalue condition $\tan \theta_v = \sigma^2 \theta_v/\mu h$. The remaining two cases are handled in exactly analogous ways. For the second case, $p/q = (1 + 1/N)^2$, the additional term can be computed using (22) and, on dividing by $\Delta t$, we get
\[ \exp\left\{-\frac{\mu^2 t}{2 \sigma^2}\right\} \frac{3 \sigma^2}{2 h^2}, \quad \mu = \frac{\sigma^2}{h}. \tag{27} \]

For the third case, $p/q > (1 + 1/N)^2$, define $\eta = (N + \frac{1}{2})\beta$. Since $\lim_{x \to 0} \cosh^{1/2} x = e^{1/2}$, so we get that $\cosh^{-1} \beta \to \exp(\sigma^2 \eta^2 t/2h^2)$. Thus, as with (24), the additional term in (19) is
\[ \frac{\Delta t \sigma^2}{h^2} \frac{\eta \exp(\sigma^2 \eta^2 t/2h^2)}{(N(1 - A) \cosh \eta \cosh \frac{1}{2} \beta - N(1 + A) \sinh \eta \sinh \frac{1}{2} \beta)}, \]

which, upon using manipulations similar to those that led to (25), becomes
\[ \frac{\eta \sinh \eta (\mu^2 h^2 - \sigma^4 \eta^2)}{\sigma^4 \eta^2 - \mu^2 h^2 + \sigma^2 \mu h}. \]

Using (22) and dividing by $\Delta t$, this additional term then becomes
\[ \sigma^2 \frac{(\mu^2 h^2 - \sigma^4 \eta^2) \eta \sinh \eta}{h^2 (\sigma^4 \eta^2 - \mu^2 h^2 + \sigma^2 \mu h)} \exp\left\{-\frac{\mu^2 t}{2 \sigma^2}\right\} \exp\left\{-\frac{\mu h}{\sigma^2}\right\} \exp\left\{-\frac{\sigma^2 \eta^2 t}{2h^2}\right\}, \quad \mu > \frac{\sigma^2}{h}. \tag{28} \]

Using (26), (27) and (28) in (19), we arrive at the continuous limit of the discrete-time density, in agreement with (8).
Appendix A. Expected value of the high, low and range

We derive the expected value of $R$. The authors have not been able to find this explicit result in the literature for the general asymmetric Brownian motion, though Feller [6] gives the result for the symmetric case. Consider the Brownian motion with an absorbing barrier at $h$. The probability density for the absorption time $\lambda$ is known (see for example [4]) and is given by the inverse Gaussian distribution

$$f_{\lambda}(t) = \frac{h}{(2\pi \sigma^2 t^3)^{1/2}} \exp\left\{\frac{(h - \mu t)^2}{2\sigma^2 t}\right\}.$$

Thus, $G_H(h) = P[H \geq h]$ is given by

$$G_H(h) = \int_0^T dt f_{\lambda}(t) = h \int_0^T \frac{dt}{t} \frac{1}{(2\pi \sigma^2 t)^{1/2}} \exp\left\{\frac{(h - \mu t)^2}{2\sigma^2 t}\right\}.$$

The expected value of $H$ is given by $E[H] = \int_0^\infty dh G_H(h)$, so we find (after a change of variables to $u = (h - \mu t)/(2\sigma^2 t)$) that

$$E[H] = \frac{1}{\sqrt{\pi}} \int_0^T dt \left(\frac{\sigma^2}{2t}\right)^{1/2} e^{-\alpha^2(t)} + \frac{\mu T}{2} + \frac{\mu}{\sqrt{\pi}} \int_0^T dt \int_0^{\alpha(t)} du e^{-u^2},$$

where $\alpha(t) = \mu t/(2\sigma^2)^{1/2}$. The following identity is useful in evaluating the second integral:

$$\int_0^A dx x \int_0^x du e^{-u^2} = \frac{\sqrt{\pi}}{4} \text{erf}(A^2) - \frac{1}{2} + \frac{A}{4} e^{-A^2}.$$

The first integral (after a change of variables) can be reduced to $(\sigma^2/\mu) \text{erf}(\alpha(T))$. The final result is then given by

$$E[H] = \frac{\mu T}{2} + \frac{\sigma}{\mu} \left[\text{erf}(\alpha)(\frac{1}{2} + \alpha^2) + \frac{\alpha e^{-\alpha^2}}{\sqrt{\pi}}\right], \quad (29)$$

where $\alpha = \alpha(T)$. Expectations for the low and the range can be obtained from the identities $E[L] = -E[H | \mu]$ and $E[R] = E[H | \mu] + E[H | -\mu]$. Using asymptotic forms for $\text{erf}(x)$ [7], we find that, when $\mu = 0$, $E[R] = 2\sqrt{2\sigma^2 T}/\pi$, reproducing the result in [6]. Asymptotically,

$$E[R] = \begin{cases} \frac{2\sigma^2}{\mu} \left(\frac{2\sigma}{\sqrt{\pi}} + \frac{2\alpha^3}{\sqrt{\pi}} + \cdots\right), & \alpha \to 0, \\ \frac{2\sigma^2}{\mu} \left(\alpha^2 + \frac{1}{2} e^{-\alpha^2} + \cdots\right), & \alpha \to \infty. \end{cases}$$

Thus, two different kinds of behaviour emerge at the different limits.

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On the maximum drawdown of a Brownian motion

References