both cases by arguments using derivatives) and a weaker version of Theorem 2(i).

With the above results in hand, the truth of C1 and C2 follows easily: C2(i) is immediate by Theorem 2(i) and since, on $(0,\infty)$, $u^p/u = u^{p-1}$ is non-decreasing when $p \geqslant 1$ and decreasing to zero when $p<1$, C1 and C2(ii) follow by Theorem 2(ii) and (iii), respectively. For C2(iii) we set $\phi(u) = u^p$, $0<p<1$, and apply (9): some routine algebra shows that $\phi(w)/w \leqslant 2\phi(x)/x$ if $(1+2x^p)^{1-p} \leqslant 2$, so from (9)

$$\tau < 1 \quad \text{if } 0 < x \leqslant x_p, \quad \text{where } x_p = \left[2^{p/(1-p)} - \tfrac{1}{2}\right]^{1/p} \tag{11}$$

Thus by (11), C2(ii) and the continuity of τ, there is a least (and a greatest) $x > x_p$ for which $\tau = 1$ and since $x_p \to \infty$ as $p \to 1$, the truth of C2 follows.

The above analysis of τ, initially motivated by a physical situation, suggests some purely mathematical questions. In C2(iii), is there *one* u_i for which $\tau = 1$? What is the general behaviour of the graph of τ in the cases $\phi(u) = u^p$? Can we draw any general conclusions about the behaviour of τ when ϕ is concave? Perhaps some computer analysis would be useful in suggesting possible answers. Asymptotic properties of τ seem easier to establish; we mention one which complements Theorem 2(iii): $\tau < 1$ for sufficiently large x if $\phi(u)/u \geqslant k > \frac{1}{2}$ for some constant k and sufficiently large u, with this result extending to $k = \frac{1}{2}$ if $\int_0^1 (1-u)/(1-\phi(u))\,du$ converges (e.g., if $\phi(u)$ has a non-zero left derivative at $u=1$). We conclude by noting that property (i) in Theorem 1 means that the projection speed always exceeds the return speed (also shown in [**1**] by conservation of total energy). This suggests the following analogous results which, at least under the hypotheses for ϕ we have used, students may wish to establish both from a physical and purely mathematical viewpoint: the descent time and ascent energy dissipation (the work done against air resistance) always exceed the ascent time and descent energy dissipation, respectively.

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A Differentiation Test for Absolute Convergence

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In this note, we describe a new test which provides a necessary and sufficient condition for absolute convergence of infinite series. The test is based solely on differentiation and is very easy to apply. It also provides a pictorial illustration for absolute convergence and divergence.

The discovery was made a few years ago when I was asked to give a lecture on infinite series to my classmates. The subject was new to us and some basic ideas were not quite appreciated at that early stage. I tried to give an informal or pictorial illustration for any concept that sounded abstract. When I mentioned that an infinite series would converge absolutely if its *far away* terms became *small enough*, I had to explain to the students (and to myself as it turned out) *how* far and *how* small.

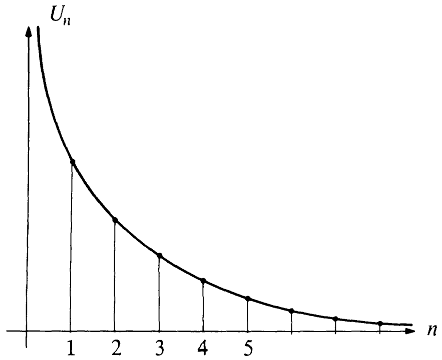

FIGURE 1

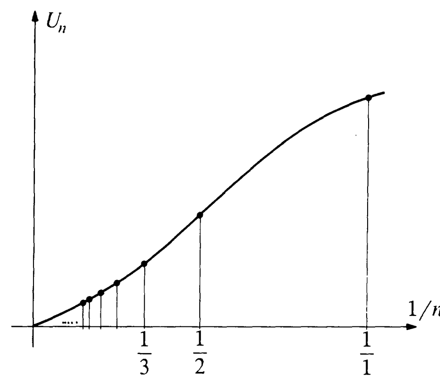

FIGURE 2

Plotting the series term by term with the summation index running along the x-axis was not very successful (FIGURE 1). I never seemed to get far enough and the terms looked quite small even with the harmonic series which, I had recently learned, was divergent. I had to come up with a picture that would show what a small term is and how it remains small if one multiplies the whole series by 10^{100} and why no matter how far away a point is, it may still not be far enough.

Remembering the duality of zero and infinity, I thought it might be nice to plot the series term by term with the *inverse* of the summation index running along the x-axis (FIGURE 2) so that we see the whole thing crowded near zero. First of all, the curve of any convergent series had to "hit" the origin since the terms go to zero as $n \to \infty$. I felt that the shape of the curve near the origin should also be related to convergence.

When I plotted divergent series like $\Sigma 1/n$ and $\Sigma 1/\sqrt{n}$, I ended up with positive or infinite slopes at the origin, but the convergent series $\Sigma 1/n^2$ had slope zero at the origin (FIGURE 3). The correspondence between the zero slope and the idea of "small terms" was appealing since a zero slope multiplied by 10^{100} is still a zero slope. The fact that the slope of a curve is a limit concept was in accordance with the "far away" idea. After the lecture, I rushed to check whether my particular illustration might generalize. After some scribbling, it gave rise to a valid criterion for absolute convergence which I call the **differentiation test**.

You probably have guessed the mechanism of the test by now. Roughly speaking, you take the infinite series in question, ΣU_n, and construct the function f defined by $f(1/n) = U_n$. First check that $f(0) = 0$. Now differentiate f and check the value of $f'(0)$: if this is also zero, the series is convergent.

Let us state this formally with the proper qualifying conditions:

DIFFERENTIATION TEST. *Let $\Sigma_{n=1}^{\infty} U_n$ be an infinite series with real terms. Let $f(x)$ be any real function such that $f(1/n) = U_n$ for all positive integers n and d^2f/dx^2 exists at $x = 0$. Then $\Sigma_{n=1}^{\infty} U_n$ converges absolutely if $f(0) = f'(0) = 0$ and diverges otherwise.*

Notice that there is a requirement that $f''(0)$ exists for the test to apply. When this requirement is satisfied, the test is guaranteed to determine whether the series is absolutely convergent or divergent. We will say more about relaxing the requirement of existence for $f''(0)$ later on, but first we present some examples to see how the test works.

We start with simple examples in which we know whether or not the series is convergent (after all, we haven't proved anything yet). Consider the two series

$$\sum_{n=1}^{\infty} \sin\frac{1}{n} \quad \text{and} \quad \sum_{n=1}^{\infty} 1 - \cos\frac{1}{n}.$$

There are many ways to verify that the first series is divergent while the second is convergent. Applying the differentiation test, we examine the functions $\sin x$ and $1 - \cos x$, both of which have second derivatives at $x = 0$. The test now tells us that the first series is divergent

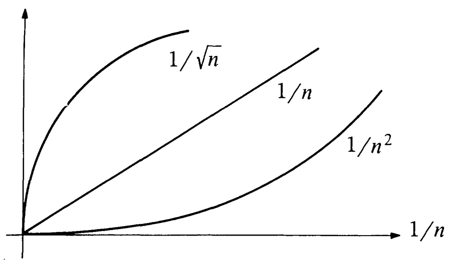

FIGURE 3

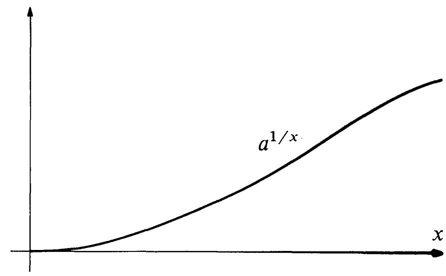

FIGURE 4

($f'(0) = 1 \neq 0$) and the second is absolutely convergent ($f(0) = f'(0) = 0$).

Another interesting example is the geometric series $\Sigma_{n=1}^{\infty} a^n$ where $0 < a < 1$. When we substitute x for $1/n$, we get the function $f(x) = a^{1/x} = e^{(\ln a)/x}$ (for $x > 0$ and zero otherwise). Since $\ln a < 0$, $f(x)$ goes to zero very quickly as $x \to 0$ (FIGURE 4). In fact, $f(x)$ has the derivatives of *all* orders at $x = 0$ equal to zero. To see this, differentiate $f(x)$ any number of times and you will always get a (finite) polynomial in $1/x$ multiplied by $f(x)$ itself. Since the exponential is "stronger" than any polynomial, all the derivatives will go to zero as $x \to 0$. This suggests that the geometric series with $0 < a < 1$ is *very* convergent, which is indeed the case.

Now we show some examples where you can determine convergence or divergence right away using the differentiation test while others will require effort to get the result. In fact, for the following examples, using other techniques to determine convergence is practically the same as writing the proof for the differentiation test in the general case.

Consider the infinite series

$$\sum_{n=1}^{\infty} \int_0^{1/n} g(t)\, dt$$

where $g(t)$ is any function that has a derivative at $t = 0$. You are required to determine the conditions on g for the series to converge absolutely. How long does it take you to conclude that it will converge absolutely if, and only if, $g(0) = 0$? To verify the result, try substituting simple functions like t^2 or e^{-t} for $g(t)$ and carry out the integration, then test the resulting series for convergence or divergence using standard methods.

Now consider:

$$\sum_{n=1}^{\infty} \sinh\left(\tanh\frac{1}{n} - \tan\frac{1}{n} + \sec\frac{1}{n^2} - \cosh\frac{1}{n}\right).$$

Since $\sinh(\tanh x - \tan x + \sec x^2 - \cosh x)$ is analytic and has zero value and zero derivative at $x = 0$, the series is absolutely convergent. You can try other compositions of simple functions like these and see that the differentiation test is equivalent to expanding the composite function $f(x)$ in a Taylor series about $x = 0$ and checking that the lowest power in the expansion is at least x^2.

If you would like to see other techniques for dealing with these examples, go through the following proof of the differentiation test which depends on such techniques.

Proof. Our proof of the differentiation test depends on L'Hospital's rule, the limit comparison test [**1**], and the integral test.

Since d^2f/dx^2 is assumed to exist at $x = 0$, we are guaranteed (among other things) that $f(x)$ is continuous at $x = 0$ and is differentiable in a neighborhood of $x = 0$ (we will need the latter to apply L'Hospital's rule). We thus have the following steps relating the conditions on $f(x)$ to the absolute convergence of $\Sigma_{n=1}^{\infty} U_n = \Sigma_{n=1}^{\infty} f(1/n)$:

(1) $f(0) = 0$ is necessary for any convergence, since $\lim_{n \to \infty} U_n = \lim_{x \to 0} f(x) = f(0)$ and if this is non-zero, the series must diverge.

(2) Suppose that $f(0)$ does equal zero, but $f'(0) = a \neq 0$. Then $\lim_{x \to 0} f(x)/x$

$= \lim_{x\to 0}(f(x)-f(0))/(x-0) = a$. Consequently, $\lim_{n\to\infty}|U_n|/(1/n) = |a| \neq 0$. By the limit comparison test, $\Sigma_{n=1}^{\infty} U_n$ diverges absolutely since the harmonic series also does.

(3) We have determined that $f(0) = f'(0) = 0$ is necessary for convergence. We now assume that this condition holds and prove sufficiency. Take $0 < u < 1$ and consider the limit

$$\lim_{x\to 0^+} \frac{f(x)}{x^{1+u}} = \lim_{x\to 0^+} \frac{f'(x)}{(1+u)x^u} = \frac{1}{1+u} \lim_{x\to 0^+} \left(\frac{f'(x)-f'(0)}{x-0}\right) x^{1-u} = \frac{f''(0)}{1+u} \lim_{x\to 0^+} x^{1-u} = 0,$$

where the first equality is an application of L'Hospital's rule. Therefore, $\lim_{n\to\infty}|U_n|/(1/n)^{1+u} = 0$ and again by the limit comparison test, $\Sigma_{n=1}^{\infty} U_n$ must converge absolutely since $\Sigma_{n=1}^{\infty} 1/n^{1+u}$ converges absolutely by the integral test.

Steps (1), (2), (3) complete the proof.

Perhaps you noticed in part (3) of the proof that the convergence did not depend critically on the existence of $f''(0)$. This is indeed the case and the existence of $f''(0)$ can be replaced by a weaker condition. We note that the condition cannot be completely removed since $\Sigma_{n=2}^{\infty} 1/n \ln n$, which is absolutely divergent by the integral test, has terms $f(1/n)$ where $f(x) = -x/\ln x$ (for $x > 0$ and zero otherwise); this function $f(x)$ has zero value and zero derivative at $x = 0$, but a non-existent second derivative.

The existence of $f''(0)$ in the differentiation test can be replaced, for example, by the existence of $\lim_{x\to 0^+} f'(x)/x^u$ or $d^2|x|^u f(x)/dx^2|_{x=0}$ for some $0 < u < 1$ (both conditions are implied by the existence of $f''(0)$ when $f(0) = f'(0) = 0$). Very minor modification of part (3) of the proof above is needed in these cases. Certain weaker conditions will also work; their discovery is left as a simple exercise.

It is also obvious that only the existence of $f'(0)$ is needed to conclude absolute divergence of a series using the test. Since divergence is seldom good news, I choose to leave the test in its simple symmetric form. Finally, one can apply the test with $f(1/n) = |U_n|$ instead of U_n. This covers complex series as well.

I should like to acknowledge Dr. Brent Smith and one of the referees for their assistance.

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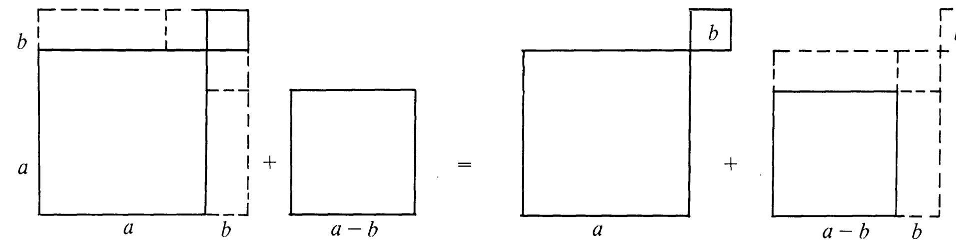
Proof without Words: Algebraic areas

$$(a+b)^2 + (a-b)^2 = 2(a^2+b^2)$$



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